Time Value of Money

jcw November, 2007

Equations copied from Wikipedia

Some standard calculations based on the time value of money are:

- **Present Value** (PV) of an amount that will be received in the future.
- **Future Value** (FV) of an amount invested (such as in a deposit account) now at a given rate of interest.
- **Present Value of an Annuity** (PVA) is the present value of a stream of (equally-sized) future payments, such as a mortgage.
- **Future Value of an Annuity** (FVA) is the future value of a stream of payments (annuity), assuming the payments are invested at a given rate of interest.
- **Present Value of a Perpetuity** is the value of a regular stream of payments that lasts "forever", or at least indefinitely.

Present value of a future sum

The present value formula is the core formula for the time value of money; each of the other formulae is derived from this formula. F example, the annuity formula is the sum of a series of present value calculations.

- The present value (PV) formula has four variables, each of which can be solved for:
  1. PV is the value at time=0
  2. FV is the value at time=n
  3. i is the rate at which the amount will be compounded each period
  4. n is the number of periods

$$PV = \frac{FV}{(1+i)^n}$$

Future value of a present sum

- The future value (FV) formula is similar and uses the same variables.

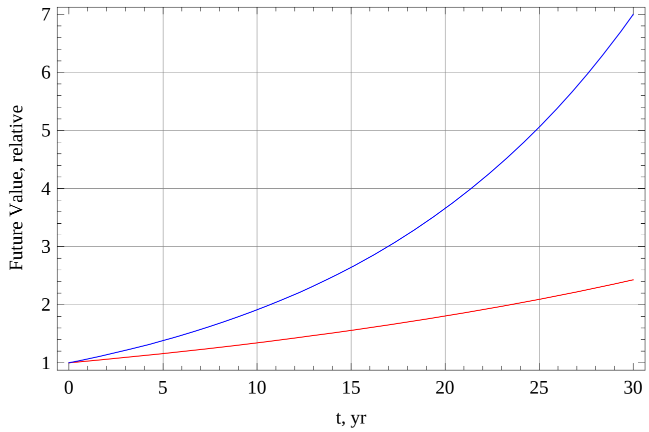
$$FV = PV \cdot (1+i)^n$$

Start out with the future value of an initial investment. The average long term yield of the stock market after inflation is 6.7%/year.

```
In[1]:= fv := pv * (1 + int / 100) ^n
```

In[2]:=
```
pv = 1.; int = 3.;
Clear[int]
Plot[{fv /. int → 3, fv /. int → 6.7}, {n, 0, 30}, Frame → True, GridLines → Automatic,
 FrameLabel → {"t, yr", "Future Value, relative"},
 PlotStyle → {Red, Blue}]
```

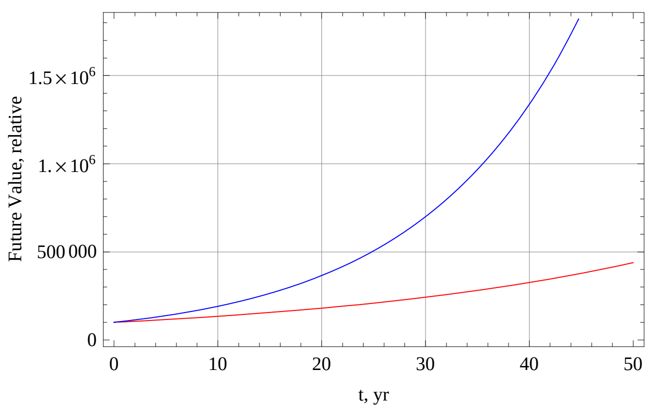
Out[4]=



Now assume I have $100,000 today from grandma Wong's death, how many years until I can retire with $1,000,000 in the bank?

In[8]:=
```
pv = 100 000.;
Clear[int]
Plot[{fv /. int → 3, fv /. int → 6.7}, {n, 0, 50}, Frame → True, GridLines → Automatic,
 FrameLabel → {"t, yr", "Future Value, relative"},
 PlotStyle → {Red, Blue}]
```

Out[10]=



Uncle Frank promises to pay me $30,000 in five year's time when he sells the farm. How much is this worth today assuming an interest rate of 6.7%?

In[11]:=
```
30 000. / (1 + 0.067) ^5
```

Out[11]= 21 692.

Since retirement options look dim, let's try the lottery! Today we win the lottery, $500,000 spread in monthly payments over 10 years. What is it really worth?

In[12]:= $$pva := \frac{a\left(1 - \frac{1}{(i+1)^n}\right)}{i}$$

In[21]:=
```
a = 1000000. / (12 * 20);
i = 0.05 / 12;
n = 12 × 20;
pva
```

Out[24]= 631355.

Future value of an annuity

- The future value of an annuity (FVA) formula has four variables, each of which can be solved for:
  1. FV(A) the value of the annuity at time=n
  2. A the value of the individual payments in each compounding period
  3. i equals the interest rate that would be compounded for each period of time
  4. n is the number of payment periods.

$$FV(A) = A \cdot \frac{(1+i)^n - 1}{i}$$

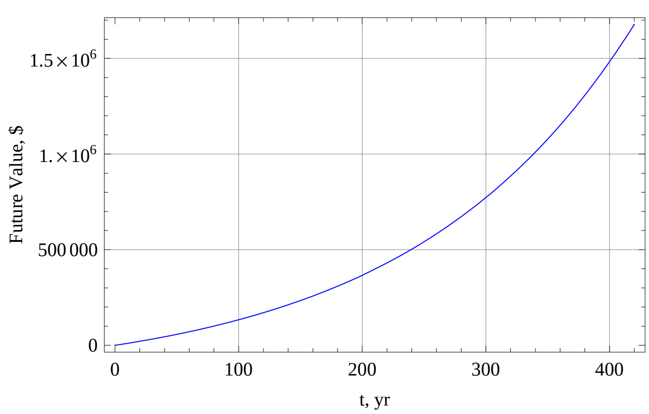
In[29]:=
$$fv := pv\,(int + 1)^n$$
$$fva := \frac{a\,((int + 1)^n - 1)}{int}$$

Jack is 30 years old and plans to put $1,000/month into his retirement account. How much money will he have when he retires at 65?

In[25]:=
```
int = 0.067 / 12;
a = 1000;
months = 35 * 12
```

Out[27]= 420

In[31]:=
```
Plot[fva, {n, 0, 420}, Frame → True, GridLines → Automatic,
 FrameLabel → {"t, yr", "Future Value, $"}, PlotStyle → {Red, Blue}]
```

Out[31]=


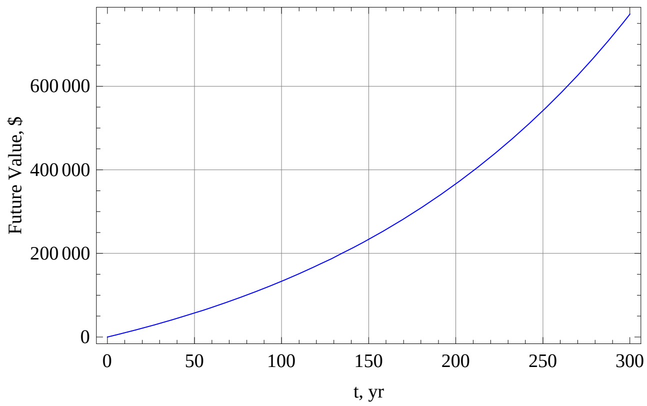
John didn't begin his retirement until reaching 40. If he puts in $1, 000/month into his retirement account. How much money will he have when he retires at 65?

In[32]:=
```
int = 0.067 / 12;
a = 1000;
months = 25 * 12
```

Out[34]= 300

In[35]:=
```
Plot[fva, {n, 0, 300}, Frame → True, GridLines → Automatic,
 FrameLabel → {"t, yr", "Future Value, $"}, PlotStyle → {Red, Blue}]
```

Out[35]=



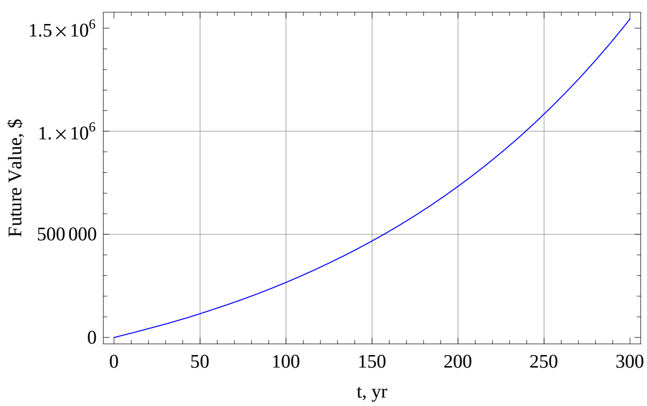
What if he puts in $2,000 / month?

In[36]:=
```
int = 0.067 / 12;
a = 2000;
months = 25 * 12
```

Out[38]= 300

In[39]:=
```
Plot[fva, {n, 0, 300}, Frame → True, GridLines → Automatic,
 FrameLabel → {"t, yr", "Future Value, $"}, PlotStyle → {Red, Blue}]
```

Out[39]=



So, even if he contributes twice as much per month, he can' make up for starting his retirement late. When will you start your retirement?

Business Problem

A new water treatment plant is to be built. Design A is old school with old technology. Capital costs are 10.34 $M and operational costs are $900,000 per year. Design B uses new technology. Capital costs are 23 $M but operational costs are only $500,000 per year. Assuming a twenty year plant life, what is the present value of the cost for each plant? Which plant will save the taxpayers money. Recent municipal bonds have gone at 5.3%.

Design A:

In[12]:= $$\text{pva} := \frac{a\left(1 - \frac{1}{(i+1)^n}\right)}{i}$$

In[40]:=
```
i = 0.053;
n = 20; a = 900 000.;
10.34 × 10^6 + pva
```

Out[42]= 2.12761×10^7

Design B:

In[43]:=
```
a = 500 000.;
23 × 10^6 + pva
```

Out[44]= 2.90756×10^7